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Harmonic monsters

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Abstract

Let Ω be a non-empty open subset of $\mathbb{R}^d$, where $d \geqslant 2$. A modern theorem on harmonic approximation is used to show that there exists a harmonic function h on Ω behaving wildly near every boundary point of Ω. The function h is analogous to the holomorphic monster functions of W. Luh.

Keywords: Universal; Harmonic; Approximation

1. Introduction

Birkhoff [4] showed that there is an entire (holomorphic) function f such that the set of translates $\{z \mapsto f(a+z) : a \in \mathbb{C}\}$ is dense in the space of all entire functions equipped with the topology of local uniform convergence. Such a function f may be thought of as behaving wildly near the boundary point ∞ of its domain $\mathbb{C}$. Luh [11–15] has undertaken a study of holomorphic functions on more general open sets that exhibit wild behaviour near every finite boundary point. The following theorem of his is proved in [14].

Theorem L. *Let Ω be a proper open subset of $\mathbb{C}$ whose components are simply connected. There exists a holomorphic function f on Ω with the following properties.*

(i) *For every finite boundary point ζ of Ω, for every compact set K with connected complement in $\mathbb{C}$, and for every continuous function g on K that is holomorphic on the*

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interior of K there exist linear transformations $\tau_n(z) = a_n z + b_n$ *with* $\tau_n(K) \subset \Omega$ *and* $\mathrm{dist}(\tau_n(K), \zeta) \to 0$ *such that* $f \circ \tau_n \to g$ *uniformly on K.*

(ii) *Further, every derivative* $f^{(j)}$ *of f and every anti-derivative of f has the boundary behaviour described in* (i).

Luh calls such functions *holomorphic monsters*. For other results on holomorphic monsters see the dissertations of Grosse-Erdmann [9] and Schneider [16], and for a survey of related topics see Grosse-Erdmann's article [10].

Dzagnidze [6] proved the following analogue of Birkhoff's theorem.

Theorem D. *There exists a harmonic function h on* $\mathbb{R}^d$, *where* $d \geqslant 2$, *such that the set of translates* $\{x \mapsto h(a + x) : a \in \mathbb{R}^d\}$ *is dense in the space of all harmonic functions on* $\mathbb{R}^d$ *equipped with the topology of local uniform convergence.*

Theorem D can be proved by using Walsh's classical theorem on harmonic approximation (see e.g. [8, Section 8.2]) or, slightly more succinctly, by using a more recent result about harmonic approximation on unbounded sets (see e.g. [2, Section 11]). The purpose of this note is to show that Theorem L also has a harmonic analogue. The key result for its proof is again a theorem on harmonic approximation, quoted below as Lemma 1.

We shall use the following notation. The Alexandroff point (at infinity) of $\mathbb{R}^d$ is denoted by ∞, and for a subset E of $\mathbb{R}^d$, we denote by ∂E the boundary of E in the one-point compactification $\mathbb{R}^d \cup \{\infty\}$ of $\mathbb{R}^d$. Let (E_n) be a sequence of non-empty bounded sets in $\mathbb{R}^d$ and let y be a point in $\mathbb{R}^d$. We write $E_n \to y$ if $\sup\{||x - y|| : x \in E_n\} \to 0$ as $n \to \infty$, where $||\cdot||$ denotes the Euclidean norm on $\mathbb{R}^d$; we write $E_n \to \infty$ if $\inf\{||x|| : x \in E_n\} \to +\infty$ as $n \to \infty$. If E is a non-empty subset of $\mathbb{R}^d$, then $\mathscr{H}(E)$ denotes the space of all functions h that are harmonic on some open set containing E. In particular, if E is open, then $\mathscr{H}(E)$ is just the space of functions that are harmonic on E. We write $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and use standard multi-index notation: if $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, then

$$D^\alpha = \partial^{\alpha_1 + \cdots + \alpha_d} / \partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}.$$

A function $\tau : \mathbb{R}^d \to \mathbb{R}^d$ will be called a *simple transformation* if $\tau(x) = ax + y$, where $a \in \mathbb{R} \backslash \{0\}$ and $y \in \mathbb{R}^d$. A simple transformation τ preserves harmonicity: $h \in \mathscr{H}(\tau(E))$ if and only if $h \circ \tau \in \mathscr{H}(E)$.

2. Main result

Theorem 1. *Let* Ω *be a non-empty open subset of* $\mathbb{R}^d$, *where* $d \geqslant 2$. *There exists a function h in* $\mathscr{H}(\Omega)$ *with the following properties.*

(i) *For every y in $\partial\Omega$, for every compact set K with connected complement in $\mathbb{R}^d$, and for every g in $\mathscr{H}(K)$ there exists a sequence (τ_n) of simple transformations with $\tau_n(K)\subset\Omega$ for each n such that $\tau_n(K)\to y$ and $h\circ\tau_n\to g$ uniformly on K.*

(ii) *Every partial derivative $D^\alpha h$ has the boundary behaviour described in* (i).

In Section 3, we will compare Theorem 1 with Theorem L and also show that in some respects Theorem 1 can be strengthened.

The following result of Gardiner (see [8, Theorem 3.11]) on tangential harmonic approximation is central to the proof of Theorem 1. In stating it, we use Ω^* to denote the one-point compactification of Ω.

Lemma 1. *Let S be a proper relatively closed subset of a connected open set Ω in $\mathbb{R}^d$ such that $\Omega^*\backslash S$ is connected and locally connected. If $H\in\mathscr{H}(S)$ and u is a positive superharmonic function on some open set containing S, then there exists a function h in $\mathscr{H}(\Omega)$ such that $0<h-H<u$ on S.*

We now prove Theorem 1. Until the final paragraph of the proof, we suppose that Ω is connected. Let $B(x,r)$ denote the open ball of centre x and radius r in $\mathbb{R}^d$; we also define $B(\infty,r)=\{y\in\mathbb{R}^d:||y||>r^{-1}\}$. For each positive integer n let C_n be a countable (finite if $\partial\Omega\backslash\{\infty\}$ is bounded) subset of Ω with the property that

$$B(y,n^{-1})\cap C_n\neq\emptyset \quad (y\in\partial\Omega). \tag{1}$$

We also suppose that the sets C_n are mutually disjoint and that every compact subset of Ω contains at most finitely many points of $\bigcup_{n=1}^{\infty} C_n = C$, say. Then for each x in C there exists a positive number r_x such that the balls $B(x,3r_x)$ are mutually disjoint and contained in Ω. Further, we choose r_x so that $r_x<n^{-1}$ when $x\in C_n$. We define

$$S=\bigcup_{x\in C}\overline{B(x,2r_x)}, \tag{2}$$

where $\bar{E}$ denotes the closure of a subset E of $\mathbb{R}^d$. For future reference, we observe that S satisfies the hypotheses of Lemma 1.

Let (P_m) be a sequence of harmonic polynomials that is dense in $\mathscr{H}(\mathbb{R}^d)$ with the topology of local uniform convergence. Let T be the countable set of ordered triples given by

$$T=\{(B(0,q),P_m,\alpha):q,m\in\mathbb{N}_0\backslash\{0\},\alpha\in\mathbb{N}_0^d\},$$

and let $((B_n,H_n,\alpha_n))$ be a sequence of elements of T in which each element of T occurs infinitely often.

We now define a certain harmonic function H on $\bigcup_{x\in C} B(x,3r_x)$. For this, we fix a point x in C and let n be the unique positive integer for which $x\in C_n$. Let τ_x be a simple transformation such that $\tau_x(B_n)=B(x,3r_x)$. Then $H_n\circ\tau_n^{-1}$ is a harmonic polynomial, and there exists a harmonic polynomial Q_x such that $D^{\alpha_n}Q_x=H_n\circ\tau_x^{-1}$ (see e.g. [1, Lemma 2]). We define H by writing $H=Q_x$ on $B(x,3r_x)$ for each x in C.

The next step is to use Lemma 1 to approximate H on the set S defined in (2) by an element of $\mathscr{H}(\Omega)$. We define a positive superharmonic function u on $\bigcup_{x\in C} B(x,3r_x)$ in the following way. If $x\in C$, then $x\in C_n$ for exactly one n and there exists a positive number A_x such that

$$\sup_{B(x,r_x)} |D^{\alpha_n} w| \leqslant A_x \sup_{B(x,2r_x)} |w| \tag{3}$$

for all bounded harmonic functions w on $B(x,2r_x)$; see e.g. [3, Corollary 1.4.3]. We define $u=(nA_x)^{-1}$ on $B(x,3r_x)$ for each x in C. Let S be the set defined in (2) and let the functions H,u be as defined above. We have already observed that the topological hypotheses of Lemma 1 are satisfied and that $H\in\mathscr{H}(S)$. Also, u is positive and superharmonic on an open set containing S. Therefore by Lemma 1, there exists a function h in $\mathscr{H}(\Omega)$ such that $0<h-H<u$ on S; that is to say, $0<h-Q_x<(nA_x)^{-1}$ on $\overline{B(x,2r_x)}$ for each x in C_n and each positive integer n. Since $D^{\alpha_n}Q_x = H_n\circ\tau_x^{-1}$ when $x\in C_n$, we find, using (3), that

$$|D^{\alpha_n}h - H_n\circ\tau_x^{-1}| < n^{-1} \quad \text{on } B(x,r_x) \quad (x\in C_n). \tag{4}$$

Now let K be a non-empty compact subset of $\mathbb{R}^d$ with connected complement, and suppose that $g\in\mathscr{H}(K), \alpha\in\mathbb{N}_0^d, y\in\partial\Omega$ and $\varepsilon>0$ are given. To complete the proof of Theorem 1 in the case where Ω is connected, it suffices to show that there is a simple transformation τ such that $\tau(K)\subset\Omega\cap B(y,\varepsilon)$ and $|(D^\alpha h)\circ\tau - g|<\varepsilon$ on K. By Walsh's harmonic approximation theorem (see e.g. [8, p. 8] or [3, p. 49]), there exists a harmonic polynomial G such that $|g-G|<\varepsilon/3$ on K. Also there exists an element P_m of the dense sequence (P_n) of harmonic polynomials such that $|P_m - G|<\varepsilon/3$ on K. Let q be an integer such that $K\subset B(0,q)$. There exist arbitrarily large values of n for which $(B(0,q),P_m,\alpha)=(B_n,H_n,\alpha_n)$, and we choose such an integer n with the property that $n>3(1+\varepsilon)/\varepsilon$. Let x be a point of $C_n\cap B(y,n^{-1})$. Then

$$\tau_x(K)\subset\tau_x(B_n)=B(x,r_x)\subset\Omega.$$

If $y\neq\infty$, then since $r_x<n^{-1}$,

$$\tau_x(K)\subset B(x,r_x)\subset B(y,2n^{-1})\subset B(y,\varepsilon);$$

similarly, $\tau_x(K)\subset B(y,\varepsilon)$ in the case where $y=\infty$. Further, by (4)

$$|(D^\alpha h)\circ\tau_x - g|<n^{-1}<\varepsilon/3 \quad \text{on } K.$$

Hence, since $H_n=P_m$

$$|(D^\alpha h)\circ\tau_x - g| \leqslant |(D^\alpha h)\circ\tau_x - H_n| + |P_m - G| + |G-g| < \varepsilon \quad \text{on } K.$$

This completes the proof in the case where Ω is connected.

Finally, we consider the general case. Let the components of Ω be Ω_j, where j belongs to some index set J. For each j in J let h_j be a harmonic monster for Ω_j; that is to say, Theorem 1 holds with Ω_j, h_j in place of Ω, h. We define a function h on Ω by putting $h=h_j$ on Ω_j for each j. Thus, $h\in\mathscr{H}(\Omega)$. Let K be a compact set such that $\mathbb{R}^d\backslash K$ is connected and suppose that $g\in\mathscr{H}(K), y\in\partial\Omega, \alpha\in\mathbb{N}_0^d$ and $\varepsilon\in(0,1)$ are given. If $y\in\partial\Omega_j$ for some j, then we choose such a j and denote it by $j(y)$ and we define

$y' = y$; otherwise, we choose an index $j(y)$ such that $B(y, \varepsilon/2) \cap \partial\Omega_{j(y)}$ is non-empty and take y' to be a point of this set. There exists a simple transformation τ such that $\tau(K) \subset \Omega_{j(y)} \cap B(y', \varepsilon/2) \subset \Omega \cap B(y, \varepsilon)$ and $|(D^\alpha h) \circ \tau - g| = |(D^\alpha h_{j(y)}) \circ \tau - g| < \varepsilon$ on K. Thus, h has the properties described in Theorem 1.

3. Improvements of Theorem 1; comparison with Theorem L

By invoking some recent results on uniform harmonic approximation, stated below as lemmas, we show how the hypotheses on the sets K and the functions g in Theorem 1 can be relaxed. For the concept of thinness, which appears in the lemmas, we refer to [3, Chapter 7]. If K is a compact subset of $\mathbb{R}^d$, then we denote by $\hat{K}$ the union of K with all bounded components of $\mathbb{R}^d \backslash K$. We denote the interior of K by K^o.

Lemma 2. *Let K be a compact subset of $\mathbb{R}^d$. The following statements are equivalent*:

(a) *for each g in $\mathscr{H}(K)$ and each positive number ε there exists a harmonic polynomial G on $\mathbb{R}^d$ such that $|g - G| < \varepsilon$ on K.*

(b) *$\mathbb{R}^d \backslash \hat{K}$ and $\mathbb{R}^d \backslash K$ are thin at the same points of K.*

Lemma 3. *Let K be a compact subset of $\mathbb{R}^d$. The following statements are equivalent*:

(a′) *for each g in $C(K) \cap \mathscr{H}(K^o)$ and each positive number ε there exists a harmonic polynomial G on $\mathbb{R}^d$ such that $|g - G| < \varepsilon$ on K.*

(b′) *$\mathbb{R}^d \backslash K$ and $\mathbb{R}^d \backslash K^o$ are thin at the same points of K.*

Lemma 2 is a refinement of Walsh's theorem and is the harmonic analogue of Runge's classical theorem on holomorphic approximation. Lemma 3 is the harmonic analogue of Mergelyan's theorem. Both lemmas are special cases of theorems of Gardiner [7] and they can also be derived from results of Bliedtner and Hansen [5]. Lemmas 2 and 3 easily yield the following improvements of Theorem 1.

Theorem 2. *Let Ω be a non-empty open subset of $\mathbb{R}^d$. There exists a function h in $\mathscr{H}(\Omega)$ with the following properties.*

(i) *For every y in $\partial\Omega$, for every compact subset K of $\mathbb{R}^d$ such that $\mathbb{R}^d \backslash \hat{K}$ and $\mathbb{R}^d \backslash K$ are thin at the same points of K and for each g in $\mathscr{H}(K)$ there exists a sequence (τ_n) of simple transformations with $\tau_n(K) \subset \Omega$ for each n such that $\tau_n(K) \to y$ and $h \circ \tau_n \to g$ uniformly on K.*

(ii) *For every y in $\partial\Omega$, for every compact subset K of $\mathbb{R}^d$ such that $\mathbb{R}^d \backslash K$ and $\mathbb{R}^d \backslash K^o$ are thin at the same points of K and for each g in $C(K) \cap \mathscr{H}(K^o)$ there exists a sequence (τ_n) as in* (i).

(iii) *Every partial derivative $D^\alpha h$ has the properties described in* (i) *and* (ii).

To prove Theorem 2, let h be as in Theorem 1, let $\alpha \in \mathbb{N}_0^d$ and let ε be a positive number. Suppose that $y \in \partial\Omega$ and that K and g are as described in Theorem 2(i). By

Lemma 2 there exists a harmonic polynomial G such that $|g - G| < \varepsilon/2$ on K. Since $\hat{K}$ is compact and $\mathbb{R}^n \setminus \hat{K}$ is connected, it follows from Theorem 1 that there exists a simple transformation τ such that $\tau(K) \subset \Omega \cap B(y, \varepsilon)$ and $|(D^\alpha h) \circ \tau - G| < \varepsilon/2$ on $\hat{K}$. Hence, $|(D^\alpha h) \circ \tau - g| < \varepsilon$ on K. This shows that $D^\alpha h$ has the approximation property described in Theorem 2(i). To show that $D^\alpha h$ has the property described in Theorem 2(ii), we use the same argument with Lemma 3 in place of Lemma 2.

It is natural to ask whether the hypotheses on K and g in Theorem 1 can be made exactly analogous to those in Theorem L: in Theorem 1 can we approximate functions g which merely belong to $C(K) \cap \mathscr{H}(K^o)$ if we assume only that K is compact and $\mathbb{R}^d \setminus K$ is connected? When $d = 2$ the answer is affirmative, and we can even relax the condition that $\mathbb{R}^2 \setminus K$ is connected.

Theorem 3. *Let Ω be a non-empty open subset of $\mathbb{R}^2$. There exists a function h in $\mathscr{H}(\Omega)$ with the following properties.*

(i) *For every y in $\partial\Omega$, for every compact set K such that $\partial K = \partial \hat{K}$, and for every g in $C(K) \cap \mathscr{H}(K^o)$ there exists a sequence (τ_n) of simple transformations with $\tau_n(K) \subset \Omega$ for each n such that $h \circ \tau_n \to g$ uniformly on K.*

(ii) *Every partial derivative $D^\alpha h$ has the boundary behaviour described in* (i).

When $d = 2$ the conditions (a), (b), (a′), (b′) of Lemmas 2, 3 and the condition $\partial K = \partial \hat{K}$ are all mutually equivalent (see [8, Corollary 1.16]), so Theorem 3 follows from Theorem 2.

Theorem 3 does not extend to higher dimensions. To see this, suppose that Theorem 3 holds in $\mathbb{R}^d$, where $d \geqslant 3$. Then, in particular, if K is a compact set in $\mathbb{R}^d$ such that $K^o = \emptyset$ and $\mathbb{R}^d \setminus K$ is connected, every function in $C(K)$ must be uniformly approximable by elements of $\mathscr{H}(K)$ and hence by harmonic polynomials. By Lemma 3, this requires that $\mathbb{R}^d \setminus K$ is nowhere thin. However, this is not necessarily true, as is shown by consideration of the case where $K = L \times [0, 1]^{d-2}$ and L is a compact subset of $\mathbb{R}^2$ such that $L^o = \emptyset$ and $\mathbb{R}^2 \setminus L$ is thin at some point (see [8, Example 1.2]).

References

[1] M.P. Aldred, D.H. Armitage, Harmonic analogues of G.R. MacLane's universal functions II, J. Math. Anal. Appl. 220 (1998) 382–395.

[2] D.H. Armitage, Uniform and tangential harmonic approximation, in: N. Arakelian, P.M. Gauthier (Eds.), Approximation, Complex Analysis, and Potential Theory, NATO Science Series, Kluwer Academic Publishers, Dordrecht, 2001, pp. 29–71.

[3] D.H. Armitage, S.J. Gardiner, Classical Potential Theory, Springer, London, 2001.

[4] G.D. Birkhoff, Démonstration d'un théorème élémentaire sur les fonctions entières, C. R. Acad. Sci. Paris 189 (1929) 473–475.

[5] J. Bliedtner, W. Hansen, Simplicial cones in potential theory II (Approximation theorems), Invent. Math. 46 (1978) 255–275.

[6] O.P. Dzagnidze, The universal harmonic function in the space E_n (Russian), Soobshch. Akad. Nauk. Gruzin. SSR 55 (1969) 41–44.

[7] S.J. Gardiner, Superharmonic extension and harmonic approximation, Ann. Inst. Fourier (Grenoble) 44 (1994) 65–91.
[8] S.J. Gardiner, Harmonic Approximation, London Mathematical Society Lecture Note Series, Vol. 221, Cambridge University Press, Cambridge, 1995.
[9] K.-G. Grosse-Erdmann, Holomorphe Monster und universelle Funktionen, Mitt. Math. Sem. Giessen 176 (1987) 1–84.
[10] K.-G. Grosse-Erdmann, Universal families and hypercyclic operators, Bull. Amer. Math. Soc. 36 (1999) 345–381.
[11] W. Luh, On universal functions, in: G. Alexits, P. Turán (Eds.), Fourier Analysis and Approximation Theory (Proc. Colloq., Budapest, 1976), Vol. II, North Holland, Amsterdam, 1978.
[12] W. Luh, Über cluster sets analytischer Funktionen, Acta Math. Acad. Sci. Hungar. 33 (1979) 137–142.
[13] W. Luh, Universalfunktionen in einfach zusammenhängenden Gebieten, Aequationes Math. 19 (1979) 183–193.
[14] W. Luh, Holomorphic monsters, J. Approx. Theory 53 (1988) 128–144.
[15] W. Luh, Multiply universal holomorphic functions, J. Approx. Theory 89 (1997) 135–155.
[16] I. Schneider, Schlichte Funktionen mit universellen Approximationseigenschaften, Mitt. Math. Sem. Giessen 230 (1997) 1–72.